I. Determine whether the relation of {$(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{R}$ **if** $\boldsymbol{x} \geq \boldsymbol{y}$} is ***reflexive***, ***symmetric***, ***antisymmetric***, ***transitive***, and/or ***a partial order***.

Solution:

1. reflexive: $\forall x \in Z^+ \; xRx$ since $x \geq x$.
2. antisymmetric: $xRy \land yRx \rightarrow x = y$ since $x \geq y \land y \geq x \rightarrow x = y$
3. transitive: $xRy \land yRz \rightarrow xRz$ since $x \geq y \land y \geq z \rightarrow x \geq z$

So, it is a partial order.

II. Let X be the set of all four-bit strings (e.g., 0011, 0101, and 1000). Define a relation R on X as $s_1 R\, s_2$ if some substring of s_1 of length 2 is equal to some substring of s_2 of length 2. Example: $0111\ R\ 1010$ (because both 0111 and 1010 contain 01). $1110\ \backsim R\ 0001$ (because 1110 and 0001 do not share a common substring of length 2). Is this relation ***reflexive***, ***symmetric***, ***antisymmetric***, ***transitive***, and/or ***a partial order***?

Solution:

1. Reflexive –$sRs\ \forall s \in X$ since whole string identical, you are able to find part of it is identical.
2. Symmetric- $If\ s_1Rs_2\ then\ s_2Rs_1$ If you find part of $s_1$is same as part of s_2, then part of s_2 is same as part of s_1
3. Not antisymmetric. Counterexample: $0011R0101 \land 0101R0011$ but $0011 \neq 0101$
4. Not transitive. Counterexample: $(0101, 0111) \in R \land (0111, 1111) \in R$ but $(0101, 1111) \notin R$

Therefore, it is not a partial order.

III. Let R_1 and R_2 be the relations on {1, 2, 3, 4} given by

$R_1 = \{(1,1), (1, 2), (3, 4), (4, 2)\}$

$R_2 = \{(1,1), (2, 1), (3, 1), (4, 4), (2, 2)\}$

List the elements of $\boldsymbol{R_1 \circ R_2}$ and $\boldsymbol{R_2 \circ R_1}$.

Solution:

$$(1,1) \in R_2 \land (1,1) \in R_1, So, (1,1) \in R_1 \circ R_2$$

$$(1,1) \in R_2 \land (1,2) \in R_1, So, (1, 2) \in R_1 \circ R_2$$

$$(2,1) \in R_2 \land (1,1) \in R_1, So, (2, 1) \in R_1 \circ R_2$$

$$(2,1) \in R_2 \land (1,2) \in R_1, So, (2, 2) \in R_1 \circ R_2$$

$$(3,1) \in R_2 \land (1,1) \in R_1, So, (3, 1) \in R_1 \circ R_2$$

$$(3,1) \in R_2 \wedge (1,2) \in R_1, So, (3,2) \in R_1{}^\circ R_2$$

$$(4,4) \in R_2 \wedge (4,2) \in R_1, So, (4,2) \in R_1{}^\circ R_2$$

Therefore, $R_1{}^\circ R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,2)\}$

Similarly, you can get the $R_2{}^\circ R_1 = \{(1,1),(1,2),(3,4),(4,1),(4,2)\}$

IV. Determine whether the given relation is an **equivalence relation** on $\{1, 2, 3, 4, 5\}$. If the relation is an **equivalence relation**, list the **equivalence classes**.

(a) $\{(\mathbf{1},\mathbf{1}),(\mathbf{2},\mathbf{2}),(\mathbf{3},\mathbf{3}),(\mathbf{4},\mathbf{4}),(\mathbf{5},\mathbf{5}),(\mathbf{1},\mathbf{3}),(\mathbf{3},\mathbf{1}),(\mathbf{3},\mathbf{4}),(\mathbf{4},\mathbf{3})\}$

(b) $\{(\boldsymbol{x},\boldsymbol{y}) \mid \mathbf{4} \textbf{ divides } \boldsymbol{x}-\boldsymbol{y}\}$

Solutions:

(a) Reflexive: $(1,1),(2,2),(3,3),(4,4),(5,5) \in R$
Symmetric: $(1,3),(3,1) \in R, (3,4),(4,3) \in R$
Since it is not transitive, it is not an equivalence relation:

$$(1,3) \in R \wedge (3,4) \in R, but\ (1,4) \notin R$$

(b) Since it is reflexive, symmetric, and transitive, it is an equivalence relation.
The equivalence relation are [1]=[5]={1, 5}, [2]={2}, [3]={3}, [4]={4}.
Please check reflexive, symmetric, and transitive yourself.

V. $R_1 = \{(x,y) \mid x + y \le 6\}$; R_1 is from *X* to *Y*;

$R_2 = \{(y,z) \mid y = z + 1\}$; R_2 is from *Y* to *Z*;

$X = Y = Z = \{1,2,3,4,5\}$

(a) Write each of the relations R_1 and R_2 in the **matrix forms**. (A_1 and A_2)

(b) Find the **matrix product $\boldsymbol{A_1A_2}$**.

(c) Use the product matric to write the **relation of $\boldsymbol{R_2 \circ R_1}$**.

Solution:

A_1	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	0
3	1	1	1	0	0
4	1	1	0	0	0
5	1	0	0	0	0

A_2	1	2	3	4	5
1	0	0	0	0	0
2	1	0	0	0	0
3	0	1	0	0	0
4	0	0	1	0	0
5	0	0	0	1	0

A_1A_2	1	2	3	4	5
1	1	1	1	1	0
2	1	1	1	0	0
3	1	1	0	0	0
4	1	0	0	0	0
5	0	0	0	0	0

$\boldsymbol{R_2 \circ R_1} =$ {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}